HOPF FIBRATION APPENDIX

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We construct a representative of the homotopy class of the Hopf map. Let D^2 denote the closed unit 2-disk, $D^2 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$. Let $C = D^2 \times [-1,1]$, that is, C is a closed solid cylinder. We begin by listing functions that will be used. For any space X, let

$$\Delta_X : X \to X \times X$$

be the diagonal map. Let $\Phi : (C, \partial C) \to (\mathbb{R}^3 \cup \{\infty\}, \infty)$ be defined by

$$\Phi(x,y,z) = \Big(\frac{x}{1-x^2-y^2}, \frac{y}{1-x^2-y^2}, \frac{z}{1-z^2}\Big)$$

for $(x,y,z) \in \operatorname{int}(C)$ and $\Phi(x,y,z) = \infty$ for $(x,y,z) \in \partial C$. Note that Φ restricted to the interior of C is a homeomorphism $\operatorname{int}(C) \to \mathbb{R}^3$. Identify $\mathbb{R}^3 \cup \{\infty\}$ with $S^3 = \{(u,v) \in \mathbb{C}^2 : \|u\|^2 + \|v\|^2 = 1\}$ by the homeomorphism $\Psi : \mathbb{R}^3 \cup \{\infty\} \to S^3$,

$$\Psi(x,y,z) = \Big(\frac{2x+2yi}{1+x^2+y^2+z^2}, \frac{2z + i(-1+x^2+y^2+z^2)}{1+x^2+y^2+z^2}\Big)$$

where $\Psi(\infty) = (0,i)$. The map Ψ is simply the inverse of the stereographic projection $S^3 \to \mathbb{R}^3 \cup \{\infty\}$, where S^3 is viewed as a subset of $\mathbb{C}^2$. Let $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the extended complex plane. Let $\psi : \widehat{\mathbb{C}} \to S^2$ be the homeomorphism

$$\psi(x+iy) = \Big(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, \frac{-1+x^2+y^2}{1+x^2+y^2}\Big).$$

Let $\mu : \widehat{\mathbb{C}} \times (\mathbb{C} \setminus 0) \to \widehat{\mathbb{C}}$ be complex multiplication where we set $\mu(\infty, z) = \infty$ for all $z \in \mathbb{C} \setminus 0$. Recall that the Hopf map $h : S^3 \to S^2$ can be defined by $h = \psi \circ h_0$, where $h_0 : S^3 \to \widehat{\mathbb{C}}$ is

$$h_0(u,w) = u/w$$

for $(u,w) \in S^3 \subseteq \mathbb{C}^2$. Define functions $h_1, h_2 : S^3 \to \widehat{\mathbb{C}}$ by

$$h_1(u,w) = u/\|w\|$$

$$h_2(u,w) = \|w\|/w$$

and where we define $h_2(u,0) = 1$. Note that h_2 is not in general continuous at points $(u,0) \in S^3$. We clearly have that

$$h = \psi \circ \mu \circ (h_1 \times h_2) \circ \Delta_{S^3}.$$

Define a reparameterization $g_1 : [-1,1] \to [-1,1]$ by

$$g_1(s) = \begin{cases} \frac{3}{2}s + \frac{1}{2} & \text{if } s \in [-1, -\frac{1}{3}] \\ 0 & \text{if } s \in [-\frac{1}{3}, \frac{1}{3}] \\ \frac{3}{2}s - \frac{1}{2} & \text{if } s \in [\frac{1}{3}, 1] \end{cases}$$

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Set $G_1 : [-1,1] \times I \to [-1,1]$ to be the homotopy $G_1(s,t) = (1-t)s + tg_1(s)$ between the identity on $[-1,1]$ and g_1. Define $H_1 : D^2 \times [-1,1] \times I \to D^2 \times [-1,1]$ by $H_1 = \mathrm{id}_{D^2} \times G_1$. Define another reparameterization $g_2 : [-1,1] \to [-1,1]$ by

$$g_2(s) = \begin{cases} -1 & \text{if } s \in [-1, -\frac{1}{3}] \\ 3s & \text{if } s \in [-\frac{1}{3}, \frac{1}{3}] \\ 1 & \text{if } s \in [\frac{1}{3}, 1], \end{cases}$$

and similarly, set $G_2(s,t) = (1-t)s + tg_2(s)$ and $H_2 = \mathrm{id}_{D^2} \times G_2$. Let $\varphi = \frac{\sqrt{5}-1}{2}$. The reason we will need this constant is due to the map $\Phi : (C, \partial C) \to (\mathbb{R}^3 \cup \{\infty\}, \infty)$. A point $(x_1, y_1, z_1) \in C$ satisfying $\sqrt{x_1^2 + y_1^2} = \varphi$ is mapped by Φ to a point (x_2, y_2, z_2) in $\mathbb{R}^3$ satisfying $\sqrt{x_2^2 + y_2^2} = 1$. Define a function $G_3 : I^2 \to I$ by

$$G_3(s,t) = \begin{cases} 0 & \text{if } s \in [0, \varphi t] \\ \frac{1}{1-t}(s - \varphi t) & \text{if } s \in [\varphi t, 1 - t + \varphi t] \\ 1 & \text{if } s \in [1 - t + \varphi t, 1] \end{cases}$$

for $t \in [0,1)$ and set

$$G_3(s,1) = \begin{cases} 0 & \text{if } s \in [0, \varphi) \\ 1 & \text{if } s \in [\varphi, 1]. \end{cases}$$

Note that G_3 is not continuous at the point $(\varphi, 1) \in I^2$ but is continuous everywhere else. We now use G_3 to define a function $H_3 : D^2 \times [-1,1] \times I \to D^2 \times [-1,1]$. For $(x,y) \in D^2$, write $(x,y) = (r\cos\theta, r\sin\theta)$. Let $z \in [-1,1]$ and $t \in I$. Define

$$H_3(r\cos\theta, r\sin\theta, z, t) = (G_3(r,t)\cos\theta, G_3(r,t)\sin\theta, z).$$

Let $H_4 : C \times I \to C$ be the composition of H_2 and H_3, that is,

$$H_4(x,t) = H_3(H_2(x,t), t)$$

where $x \in C$ and $t \in I$. Let $h_1' = h_1 \circ \Psi \circ \Phi$ and $h_2' = h_2 \circ \Psi \circ \Phi$. Define $H : (C \times I, \partial C \times I) \to (S^2, (0,0,-1))$ by

$$H = \psi \circ \mu \circ (h_1' \times h_2') \circ (H_1 \times H_4) \circ \Delta_{C \times I}.$$

We will show H is continuous, in which case it defines a homotopy between $h \circ \Psi \circ \Phi$ (which we are identifying with the Hopf map) and $g = H(x,1) : (C, \partial C) \to (S^2, (0,0,-1))$, the map described in the blog post.

Proposition 0.1. *The function H is continuous.*

Proof. In the defintion of H, the only functions which are not continuous are H_4 and h_2'. The function H_4 is not necessarily continuous at points in

$$B_1 = \{(x,y,z,1) \in C \times I : \sqrt{x^2 + y^2} = \varphi\}$$

and h_2' is not necessarily continuous at points

$$B_2 = \{(x,y,0,t) \in C \times I : \sqrt{x^2 + y^2} = \varphi\}.$$

Therefore it suffices to check continuity of H at points in B_1 and at points $a \in C \times I$ such that $H_4(a) \in B_2$. However, $\{a \in C \times I : H_4(a) \in B_2\} \subset B_2$, hence it suffices to check continuity of H at exactly $B_1 \cup B_2$. To see that H is continuous at points in B_2, note that if $x \in B_2$, then $h_1' \circ H_1(x) = \infty \in \widehat{\mathbb{C}}$. Since $h_1' \circ H_1$ is continuous, and h_2' has image in the unit circle $\{z \in \widehat{\mathbb{C}} : \|z\| = 1\}$, hence is bounded away from 0, we have that

$$\mu \circ (h_1' \times h_2') \circ (H_1 \times H_4) \circ \Delta_{C \times I}$$

is continuous at points in B_2, and thus so is H.

Write $B_1 = B_1' \cup B_1''$ where

$$B_1' = \{(x, y, z, 1) \in C \times I : \sqrt{x^2 + y^2} = \varphi,\ z \in [-1/3, 1/3]\}$$

$$B_1'' = \{(x, y, z, 1) \in C \times I : \sqrt{x^2 + y^2} = \varphi,\ z \in [-1, -1/3) \cup (1/3, 1]\}$$

We have that $h_1' \circ H_1$ maps points of B_1' to $\infty \in \widehat{\mathbb{C}}$, and so completely analogous to the case of B_2, H is continuous at points in B_1'. Lastly, if $(x, y, z, 1) \in B_1''$, let $U \subset D^2 \times [-1, -1/3) \cup (1/3, 1]$ be a neighborhood of (x, y, z) in C. Let N_ε be the set of points in C whose distance to ∂C is less than ε. Then for any $\varepsilon > 0$, due to the homotopy H_2, there exists $\delta > 0$ so that H_4 maps $U \times (1 - \delta, 1]$ into N_ε. Since $\Psi \circ \Phi$ maps N_ε to an open neighborhood of the basepoint $(0, i)$ of $S^3 \subseteq \mathbb{C}^2$, and h_2 is continuous in a neighborhood of $(0, i)$, we have that $h_2' \circ H_4$ is continuous at points in B_1''. Hence H is continuous at points in B_1''. □